Introduction to Dynamical Systems
Mock Exam Solutions 2025

Problem 1.

(i) Let (X, d) be a metric space, and $T : X \to X$ continuous. Define what it means for T to be transitive, or to be minimal.

(ii) Consider $X = S^1 \times S^1$ equipped with the canonical metric induced from $\mathbb{R}^2$, and let $T : X \to X$ be defined by

$$T(x, y) = \left(x^2, y^3\right).$$

Is T transitive? Is T minimal? Provide detailed arguments for your answers.

Solution. Part (i) consists of two definitions. A map T as in the statement is transitive provided that there exists at least one $x \in X$ whose forward orbit by T,

$$\mathcal{O}^+(x) := \left\{T^j(x) \colon j \in \mathbb{N}\right\},$$

is dense in X, $\overline{\mathcal{O}^+(x)} = X$. It is minimal if this is satisfied for every element.

For part (ii), notice that T is not minimal, since by interpreting T on $[0, 1] \times [0, 1]$ via $T(x, y) = (2x, 3y)$ modulo 1, the origin $(0, 0)$ does not give rise to a dense orbit. On the other hand, it is transitive. Indeed, appealing to the Birkhoff transitivity theorem, the space X is complete and has a countable basis of open sets, hence it remains to show that given any $U, V \subset X$ open there is $n \in \mathbb{N} \cup \{0\}$ with the property that

$$T^n(U) \cap V \neq \emptyset. \tag{1}$$

This is easy to check knowing that the doubling and tripling maps (T component-wise) are both topologically mixing. That is, for any given open sets $U, V \subset X$, there is $N \in \mathbb{N} \cup \{0\}$ such that for all $n \geq N$, (1) holds true.

To show this, we may reduce to considering an open interval $U = (a, b)$. Then, if N is large enough, the doubling map $T_2(x) = 2x$ satisfies $T_2^N(U) = S^1$, as $2^N(b - a) > 1$. Hence $T_2^N(U) \cap V \neq \emptyset$. The same argument works for the tripling map $T_3(x) = 3x$.

It remains to check that products preserve the mixing property. To this end, we may let $U_1, U_2 \subset S^1$ be nonempty, open intervals in the topological basis of S^1. (Their products construct the basis of the topology in X.) This fixes N_2, N_3 for which $T_2^{N_2}(U_1) = T_3^{N_2}(U_2) = S^1$, and it suffices to let $N > N_2, N_3$ to deduce that $T^N(U_1 \times U_2) = X$. Hence for any nonempty open set $V \subset X$, (1) holds trivially for $n \geq N$. One then finally concludes by applying the Birkhoff transitivity theorem (**Theorem 2.1** in Lecture2.pdf, stated below). $\square$

Problem 2. Let $T:\mathbb{R}\to\mathbb{R}$ be given by $T(x)=\frac{1}{2}(x-x^{-1})$ if $x\neq 0$ and by $T(0)=0$.

(i) Show that if μ is the measure given by

$$\mu(A)=\pi^{-1}\int_A(1+x^2)^{-1}\,dx,$$

then T is measure preserving.

(ii) Show that there is a zero measure set $\mathcal{N}\subset\mathbb{R}_+$ such that for all $x\in\mathbb{R}_+\backslash\mathcal{N}$, there are infinitely many distinct n_j such that

$$T^{n_j}(x)\in(0,1).$$

Solution. Part (i) follows from a direct computation. Setting $y=T(x)$, we compute to find

$$x=y\pm\sqrt{y^2+1},$$

and given any $A\subset\mathbb{R}$, we may split $T^{-1}(A)$ into its negative and non-negative parts, and

$$\begin{aligned}\int_{T^{-1}(A)}\frac{1}{1+x^2}\,\mathrm{d}x&=\int_{[T^{-1}(A)\cap(-\infty,0)]}\frac{1}{1+x^2}\,\mathrm{d}x+\int_{[T^{-1}(A)\cap[0,\infty)]}\frac{1}{1+x^2}\,\mathrm{d}x\\&=\int_A\frac{2y^2+1+2y\sqrt{y^2+1}}{2\left[y^2+y\sqrt{y^2+1}+1\right]^2}\,\mathrm{d}y+\int_A\frac{2y^2+1-2y\sqrt{y^2+1}}{2\left[y^2-y\sqrt{y^2+1}+1\right]^2}\,\mathrm{d}y\\&=\int_A\frac{1}{1+y^2}\,\mathrm{d}y.\end{aligned}$$

We conclude that $\pi\mu(T^{-1}(A))=\pi\mu(A)$ and part (i) follows. Part (ii) can be approached through a Poincaré recurrence. Define

$$A_0:=(0,1)\cap\limsup_{n\to\infty}T^{-n}((0,1)),\quad B_0=(0,1)\setminus A_0.$$

Poincaré recurrence now implies that

$$\mu(A_0)=\mu((0,1))=\pi^{-1}\int_0^1\frac{1}{1+x^2}\,\mathrm{d}x=\pi^{-1}\arctan(1)=1/4,\quad \mu(B_0)=0.$$

To extend this to $\mathbb{R}_+$ we observe that $T(x)<(2/3)\cdot x$ whenever $x\geq 1$, meaning that iterations of T satisfy $T^k(x)<(2/3)^k\cdot x$, so for any $x\in[1,\infty)$ there is $n_x\in\mathbb{N}$ such that $T^{n_x}(x)\in(0,1)$. If $x\in A_1$, where

$$A_1:=\{x\in[1,\infty)\colon T^n(x)\in(0,1)\text{ for infinitely many }n\},\quad B_1:=[1,\infty)\setminus A_1,$$

then $T^{n_x}(x)\in A_0$.

Notice that each $x\in B_1$ eventually lands in B_0 via T^{n_x}, since $T^{n_x}(x)\in(0,1)=B_0\sqcup A_0$, and $T^{n_x}(x)\in A_0$ would contradict $x\in B_1$. Hence $x\in T^{-n_x}(B_0)$ and consequently

$$B_1\subset\bigcup_{n\geq 1}T^{-n}(B_0).$$

However, each $T^{-n}(B_0)$ has measure zero, since $\mu(B_0)=0$ and T is measure preserving, so

$$\mu(B_1)\leq\sum_{n\geq 1}\mu(T^{-n}(B))=\sum_{n\geq 1}\mu(B)=0.$$

We deduce that B_1 must be of measure zero, and A_1 must accumulate full measure in $[1,\infty)$. We conclude by letting $\mathcal{N}:=B_0\sqcup B_1$, and notice that $\mathbb{R}_+\setminus\mathcal{N}=A_0\sqcup A_1$, exactly as in the statement. □

Problem 3. Show that if (X, m) is a probability space and $T : X \to X$ is measurable and measure preserving, then T is ergodic if and only if 1 is a *simple* eigenvalue for the map

$$U_T : L^2(X, m) \to L^2(X, m)$$

given by

$$U_T f = f \circ T.$$

Solution. Begin by assuming that T is ergodic. Then every time $A \subset X$ is T-invariant, it must happen that $m(A) \in \{0, 1\}$. That 1 is a simple eigenvalue for T means that it is attached to a subspace of the Hilbert space L^2 of dimension one, i.e. one that is generated by a single function. Since the constant functions c satisfy $U_T c = c \circ T = c$, this is the subspace attached to the eigenvalue 1.

Assume for contradiction that there is an a.e. non-constant $f \in L^2(X, m)$ satisfying $f(T(x)) = f(x)$. Then, consider the sets $A_c = \{x \in X \colon f(x) \geq c\}$. Since f is non-constant, there is c_* for which $A \coloneqq A_{c_*}$ is of measure $0 < m(A) < 1 = m(X)$. And moreover, $x \in A$ happens if and only if $f(T(x)) = f(x) \geq c$, the same as $T(x) \in A$. Hence $T^{-1}(A) = A$. But since T is ergodic, $m(A) \in \{0, 1\}$ sees a contradiction.

Conversely, if T were not ergodic, then there is an invariant set A for which it must happen that $m(A) \in (0, 1)$. By just setting $f = \mathbb{1}_A$, we see that f is non-constant, since it takes two different values on two sets of positive measure. $\square$

Problem 4. Let $A \in \mathrm{Mat}(n \times n, \mathbb{R})$ a real invertible square matrix, which is hyperbolic, and let

$$\phi(x) = Ax + g(x),\ g \in C^1(\mathbb{R}^n; \mathbb{R}^n)$$

a C^1-diffeomorphism which leaves the origin fixed. State (but do not prove) the Hadamard-Perron theorem. Show that there is a decreasing sequence of open neighborhoods U_j, $j \geq 1$ of the origin, such that

$$\cap_{j=1}^{\infty} U_j = \{0\},$$

and such that the following holds: for $j < k$ and arbitrary $x \in U_j$, either $\phi^l(x) \in U_j$ for all $l \geq 0$, or else there is some $l_* > 0$ such that

$$\phi^{l_*}(x) \notin U_k.$$

You may refer to the proof of Hadamard-Perron.

Solution. Since A is hyperbolic, there is a splitting $\mathbb{R}^n = E_- \oplus E_+$ into the stable and unstable spaces with respect to A. Writing $A_\pm = A_{|E_\pm}$, we let $\|\cdot\|$ be the norm with respect to which A_+, A_-^{-1} are contracting, and introduce

$$Q_r := \{x = (x_+, x_-) \in E_+ \oplus E_- \colon\ \|x_\pm\| \leq r\}, \quad r > 0$$

and the stable and unstable manifolds,

$$W_+^{\mathrm{loc}}(Q_r) := \left\{x \in Q_r \colon \phi^j(x) \in Q_r\ \forall j \geq 0\right\},$$
$$W_-^{\mathrm{loc}}(Q_r) := \left\{x \in Q_r \colon \phi^{-j}(x) \in Q_r\ \forall j \geq 0\right\}.$$

The Hadamard-Perron theorem reads as follows.

Theorem (Hadamard-Perron)**.** *Assuming r to be small enough (depending on ϕ), there is a Lipschitz continuous function $h_+ \colon E_+ \longrightarrow E_-$, with $h_+(0) = 0$, such that*

$$W_+^{\mathrm{loc}}(Q_r) = \left\{x \in Q_r \colon \phi^j(x) \in Q_r\ \forall j \geq 0,\ \phi^j(x) \underset{j\to\infty}{\to} 0\right\}$$
$$= \{x \in Q_r \colon x = (x_+, h_+(x_+)) \in E_+ \oplus E_-\}$$

Thus $W_+^{\mathrm{loc}}(Q_r)$ is given by the graph of a Lipschitz function over $E_+ \cap Q$. An analogous result holds for $W_-^{\mathrm{loc}}(Q_r)$.

Finally, set $U_j = \mathring{Q}_{r_j}$, where $r_j \to 0$ as $j \to \infty$ is a strictly decreasing sequence of positive real numbers defined for $j \in \mathbb{N}$ and r_1 is small enough such that δ in the proof of Hadamard-Perron is small enough so that the conclusion of the proof holds. Then

$$\cap_{j=1}^{\infty} U_j = \{0\}$$

holds trivially, since $x \in \cap_{j=1}^{\infty} U_j$ implies that $\|x\| < r_j$ for each $j \geq 1$, so $\|x\| = 0$ and therefore $x = 0$. Letting $k \geq 2$ and $x \in U_j$ for $j < k$, and assuming that $\phi^l(x) \in U_j$ for all $j < k$, arbitrary $x \in U_j$ and all $l \geq 0$ is false, we deduce that there is l_* for which $\phi^{l_*}(x) \notin U_j$ for some $j < k$. Since $U_k \subset U_j$ because the sequence is decreasing, it must happen that $\phi^{l_*}(x) \notin U_k$. $\square$

Problem 5.

(i) Formulate (without proof) the Hartman-Grobman theorem in arbitrary dimension.

(ii) Give an example of a C^1-diffeomorphism $\phi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ with hyperbolic fixed point $x_* = 0$ for which there is no C^1-diffeomorphism $h : U \to V$ with U, V open neighborhoods of 0, and such that

$$h^{-1} \circ \phi \circ h = D\phi(0)$$

on U. Provide a detailed argument.

Solution. For part (i) we first define what a hyperbolic matrix is. We say that an invertible matrix with real entries $A \in \mathrm{Mat}(n \times n, \mathbb{R})$ is hyperbolic if all of its complex eigenvalues λ satisfy $|\lambda| \neq 1$.

Theorem (Hartman-Grobman)**.** *Let $\phi\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ a C^1-diffeomorphism, for which $x_0 = 0$ is a hyperbolic fixed point, i.e. $\phi(0) = 0$ and $D\phi(0) \in \mathrm{Mat}(n \times n; \mathbb{R})$ is hyperbolic. Then there exist neighbourhoods U, V of 0 and a homeomorphism $h\colon U \longrightarrow V$ such that $h(0) = 0$ and*

$$h \circ \phi(x) = A \circ h(x),$$

provided both $x, \phi(x) \in U$. It follows that

$$h \circ \phi^j \circ h^{-1}(x) = A^j x,$$

provided $x \in V$ and $\phi^k(h^{-1}(x)) \in U$ for $k = 1, \dots, j$.

Part (ii) can be approached through a contradiction argument. For instance, the system

$$\phi(x,y) = \begin{pmatrix} \lambda x \\ \lambda^{-1} y + x^2 \end{pmatrix}, \quad A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad 0 < \lambda < 1,$$

satisfies

$$D\phi(x,y) = \begin{pmatrix} \lambda & 0 \\ 2x & \lambda^{-1} \end{pmatrix}, \quad D\phi(x,y)^2 = \begin{pmatrix} \lambda^2 & 0 \\ 2x(\lambda + \lambda^{-1}) & \lambda^{-2} \end{pmatrix},$$

and if we denote $[D\phi(x,y)^n]_{21} = 2x r_n$, then we obtain a recurrence starting at $r_1 = 1$ that satisfies $r_{n+1} = \lambda r_n + \lambda^{-n}$. By computing the first few terms one can easily conclude by induction that

$$r_n = \lambda^{-(n-1)} \sum_{j=0}^{n-1} \lambda^{2j} = \lambda^{-(n-1)} \frac{1 - \lambda^{2n}}{1 - \lambda^2}.$$

Now, the conjugacy implies that

$$Dh(\phi^n(x,y)) \cdot D\phi(x,y)^n = D\phi(0)^n \cdot Dh(x,y). \tag{2}$$

Notice that by restricting to a small neighborhood V of the origin, we may write

$$Dh(x,y) = \begin{pmatrix} a(x,y) & b(x,y) \\ c(x,t) & d(x,y) \end{pmatrix}, \quad (x,y) \in V,$$

where a, b, c, d are continuous functions, given that h is C^1. Moreover, by restricting ourselves to a small closed ball B_V centered at the origin and contained in V, and since the determinant of $Dh(x,y)$ is a continuous function of a, b, c and d, we deduce that it must be bounded from below by a positive constant $\varepsilon > 0$, as otherwise the invertibility of Dh would be violated. I.e., for all $(x, y) \in B_V$,

$$\varepsilon < |\det Dh(x, y)| = |ad - bc|.$$

At the origin, $Dh(0, 0)$ is a diagonal matrix (which follows from (2)), $Dh(0, 0) = \mathrm{diag}(a_0, d_0)$, and moreover $a_0, d_0 \neq 0$ since $Dh(0, 0)$ must be invertible. For an initial point (x_0, y_0) we denote

$$Dh(\phi^n(x_0, y_0)) = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}.$$

Then from (2) we deduce

$$a_n \lambda^n + 2x_0 r_n b_n = \lambda^n a_0, \quad c_n \lambda^n + 2x_0 d_n r_n = 0,$$

which can be rewritten as

$$d_n = -\frac{\lambda^n c_n}{2x_0 r_n}, \quad a_n = a_0 - \frac{2x_0 r_n b_n}{\lambda^n}.$$

Plugging these into the determinant results in

$$\varepsilon < \left| a_0 \left(\frac{\lambda^n c_n}{2x_0 r_n} \right) \right| = \left| \frac{a_0}{2x_0} \right| \cdot \left| \frac{1 - \lambda^2}{\lambda} \right| \cdot \left| \frac{\lambda^{2n}}{1 - \lambda^{2n}} \right| = C(a_0, x_0, \lambda) \left| \frac{\lambda^{2n}}{1 - \lambda^{2n}} \right|.$$

Now, notice that the rightmost term converges to zero as $n \to \infty$. Let now (x_0, y_0) be a point in the intersection of the stable manifold with B_V. Then, $\phi^n(x_0, y_0) \to (0, 0)$ as $n \to \infty$, which extends the validity of the inequality above to all $n \in \mathbb{N}$. Letting N be large enough such that the right hand side above becomes smaller than ε, we reach a contradiction. We therefore conclude that h cannot be C^1. □

Problem 6.

(i) State and prove the Poincaré recurrence theorem.

(ii) Let (X, d) be a metric space which comes equipped with a finite measure m, such that all open sets U are measurable. Assume that (X, d) admits a countable base. Show that for almost all $x \in X$, we have

$$\liminf_{n\to\infty} d(x, T^n x) = 0.$$

Solution. Part (i) simply requires the following statement together with its proof.

Theorem (Poincaré recurrence)**.** *Let (X, m) be a finite measure space, and $T\colon X \longrightarrow X$ a measure preserving map (i.e. $m(T^{-1}(A)) = m(A)$ for any measurable set A). If*

$$A_0 = A \cap \limsup_{n\to\infty} T^{-n}(A),$$

then

$$m(A_0) = m(A).$$

Proof. We observe that

$$A_0^c = \bigcup_{n=1}^{\infty} C_n, \quad C_n = \left\{x \in A\colon T^j(x) \notin A\ \forall j \geq n\right\}.$$

It suffices to show that $m(C_n) = 0$ for all $n \geq 1$. Note that

$$C_n = A \setminus \bigcup_{j\geq n} T^{-j}(A).$$

We also have the inclusion

$$C_n \subset \bigcup_{j\geq 0} T^{-j}(A) \setminus \bigcup_{j\geq n} T^{-j}(A).$$

We therefore infer that

$$m(C_n) \leq m\left(\bigcup_{j\geq 0} T^{-j}(A)\right) - m\left(\bigcup_{j\geq n} T^{-j}(A)\right),$$

but since

$$T^{-n}\left(\bigcup_{j\geq 0} T^{-j}(A)\right) = \bigcup_{j\geq n} T^{-j}(A)$$

and T is measure preserving, we conclude that

$$m(C_n) = m\left(\bigcup_{j\geq 0} T^{-j}(A)\right) - m\left(\bigcup_{j\geq 0} T^{-j}(A)\right) = 0, \quad n \geq 1,$$

which concludes the proof. □

For part (ii), given $\varepsilon > 0$ we may choose a countable covering of X by open (and therefore measurable) balls of radius $\epsilon = \varepsilon/2$, $X = \cup_{n\geq 1} B_n^\epsilon$, where $B_n^\epsilon = B(x_n, \epsilon)$. Applying the Poincaré recurrence to each B_n^ϵ, we find that

$$m(B_n^\epsilon \cap \liminf_{k\to\infty} T^{-k}(B_n^\epsilon)) = m(B_n^\epsilon),$$

so for almost all $x \in B_n^\epsilon$, $T^k(x) \in B_n^\epsilon$ for infinitely many k. This allows us to estimate

$$d(x, T^k(x)) \leq d(x, x_n) + d(x_n, T^k(x)) \leq 2\epsilon = \varepsilon.$$

Therefore for almost all $x \in B_n^\epsilon$, $\liminf_{k\to\infty} d(x, T^k(x)) \leq \varepsilon$. Define now

$$N^\epsilon := \left\{ x \in X \colon \liminf_{k\to\infty} d(x, T^k(x)) > \varepsilon \right\}, \quad N_n^\epsilon = N \cap B_n^\epsilon.$$

Then,

$$m(N^\epsilon) = m\left(N^\epsilon \cap \bigcup_{n\geq 1} B_n^\epsilon \right) \leq \sum_{n\geq 1} m(N_n^\epsilon),$$

but since

$$m(B_n^\epsilon) = m(B_n^\epsilon \cap \liminf_{k\to\infty} T^{-k}(B_n^\epsilon)) + m(N_n^\epsilon),$$

we conclude that $m(N_n^\epsilon) = 0$ and therefore $m(N^\epsilon) = 0$. Letting $\varepsilon \to 0$ we conclude, by continuity of the measure m, that the set of points satisfying $\liminf_{k\to\infty} d(x, T^{-k}(x)) \neq 0$ is of measure zero, since the N^ϵ form a sequence of chained sets as $\varepsilon \to 0$ and their measure is bounded by that of X. $\square$

Problem 7.

(i) Formulate and prove the von Neumann mean ergodic theorem.

(ii) Let (X, m) be a finite measure space, and let $T : X \to X$ be measurable and measure preserving. Show that if $f \in L^3(X, m)$, then the limit

$$\lim_{N\to+\infty} \frac{1}{N} \sum_{j=0}^{N-1} f(T^j x) =: f_*(x)$$

exists in the L^3-sense.

Solution. We begin part (i) by introducing a few concepts. First, given (X, m) a finite measure space, and $T\colon X \longrightarrow X$ a measure preserving map, we define the operator $U_T f := f \circ T$ and the subspace V of T-invariant L^2 functions

$$V := \left\{ g \in L^2(X, m) \colon U_T g = g \right\}.$$

Finally, given the structure of $L^2(X, m)$ as a Hilbert space, we define P_T to be the orthogonal projection onto V.

Theorem (Von Neumann Mean Ergodic Theorem)**.** *Let (X, m) be a finite measure space and $T\colon X \longrightarrow X$ a measure preserving map. Then for any $f \in L^2(X, m)$,*

$$\lim_{N\to\infty} \left\| \frac{1}{N} \sum_{j=0}^{N-1} f \circ T^j - P_T f \right\|_{L^2} = 0.$$

Moreover, if T is ergodic, then

$$\lim_{N\to\infty} \left\| \frac{1}{N} \sum_{j=0}^{N-1} f \circ T^j - \frac{1}{m(X)} \int_X f \,\mathrm{d}m \right\|_{L^2} = 0.$$

Proof. Begin by defining

$$W := \overline{\{f - U_T f \colon f \in L^2(X, m)\}}.$$

Claim

It holds that $V = W^\perp$. Indeed, if $g \in V$, then

$$\langle g, f - U_T f\rangle = \langle g, f\rangle - \langle g, U_T f\rangle = \langle g, f\rangle - \langle U_T g, U_T f\rangle = \langle g, f\rangle - \langle g, f\rangle = 0,$$

and $\{f - U_T f \colon f \in L^2(X, m)\}$ is dense in W. Hence $V \subset W^\perp$. On the other hand, assuming that

$$\langle g, f - U_T f\rangle = 0 \quad \forall f \in L^2(X, m),$$

one deduces that $g - U_T^* g = 0$, but then

$$0 = \|g - U_T^* g\|_{L^2}^2 = 2\,\|g\|_{L^2}^2 - 2\mathrm{Re}\,\langle U_T g, g\rangle.$$

The equality case for the Cauchy-Schwarz inequality then implies that $g \in V$, so $W^\perp \subset V$.

In view of the claim, we need to verify the limiting relation in the statement only in the cases $f \in V$ and $f \in W$. The first one is direct, since then $U_T f = f$, and therefore $f \circ T^j = f$, but then the averages are all $f = U_T f$.

The second case is slightly more delicate. Assume $f \in W$ and first consider $f = g - U_T g$ for some $g \in L^2(X, m)$. Then,

$$f \circ T^j = g \circ T^j - g \circ T^{j+1}, \quad j \geq 0.$$

This implies that we arrive at a telescopic sum

$$\frac{1}{N}\sum_{j=0}^{N-1} f \circ T^j = \frac{1}{N}\left(g - g \circ T + g \circ T - g \circ T^2 + \cdots + g \circ T^{N-1} + g \circ T^N\right) = \frac{1}{N}(g - g \circ T^N).$$

We conclude that

$$\left\|\frac{1}{N}\sum_{j=0}^{N-1} f \circ T^j\right\|_{L^2} \leq \frac{1}{N}\|g\|_{L^2},$$

which converges to 0, and $P_T f = 0$ as well.

More generally, if $f \in W$ is arbitrary, then there is a sequence of $g_i \in L^2(X, m)$ such that

$$\lim_{i \to \infty} \|f - (g_i - U_T g_i)\|_{L^2} = 0.$$

Therefore, given any $\varepsilon > 0$ there is i_0 for which $\|f - (g_{i_0} - U_T g_{i_0}\|_{L^2} < \varepsilon$. Picking N large enough so that

$$\left\|\frac{1}{N}\sum_{j=0}^{N-1}(g_{i_0} - U_T g_{i_0})\right\|_{L^2} < \varepsilon.$$

We are then able to conclude that

$$\left\|\frac{1}{N}\sum_{j=0}^{N-1} f \circ T^j\right\|_{L^2} \leq \left\|\frac{1}{N}\sum_{j=0}^{N-1}(f - (g_{i_0} - U_T g_{i_0})) \circ T^j\right\|_{L^2} + \left\|\frac{1}{N}\sum_{j=0}^{N-1}(g_{i_0} - U_T g_{i_0}) \circ T^j\right\|_{L^2} \leq 2\varepsilon,$$

from which the result follows since $\varepsilon > 0$ is arbitrary. □

Part (ii) is a particular case of **Exercise 2** in Problem Set 5, and we have to exploit the density in L^p of L^∞. Indeed, for $p \geq 2$, given $f \in L^p(X, m)$, we may consider $f_\lambda = f \mathbb{1}_{\{|f| < \lambda\}}$ for $\lambda > 0$, to see that $f_\lambda \in L^\infty$. Then

$$\|f - f_\lambda\|_{L^p} = \|f \cdot \mathbb{1}_{\{|f| \geq \lambda\}}\|_{L^p} \longrightarrow 0, \ \lambda \to \infty.$$

Notice too that given $f \in L^\infty$,

$$\|f\|_{L^p} \leq \|f\|_{L^\infty}^{(p-2)/p} \|f\|_{L^2}^{2/p}. \tag{3}$$

We aim to show that the sequence of averages is a Cauchy sequence in $L^3(X, m)$. To this end, first let $\varepsilon > 0$ and assume that $f \in L^p \cap L^\infty$. Then, since the limit does exist in the L^2-sense by the Von Neumann Ergodic Theorem, the sequence is Cauchy in L^2 and we may find N, M such that

$$\left\|\frac{1}{N}\sum_{j=0}^{N-1} f \circ T^{-j} - \frac{1}{M}\sum_{j=0}^{M-1} f \circ T^{-j}\right\|_{L^2}^{2/p} \leq \frac{\varepsilon}{1 + (2\|f\|_{L^\infty})^{(p-2)/p}}.$$

Then, using (3) we reach

$$\begin{aligned}
&\left\| \frac{1}{N} \sum_{j=0}^{N-1} f \circ T^{-j} - \frac{1}{M} \sum_{j=0}^{M-1} f \circ T^{-j} \right\|_{L^p} \\
&\leq \left\| \frac{1}{N} \sum_{j=0}^{N-1} f \circ T^{-j} - \frac{1}{M} \sum_{j=0}^{M-1} f \circ T^{-j} \right\|_{L^2}^{2/p} \left\| \frac{1}{N} \sum_{j=0}^{N-1} f \circ T^{-j} - \frac{1}{M} \sum_{j=0}^{M-1} f \circ T^{-j} \right\|_{L^\infty}^{(p-2)/p} \\
&\leq \frac{\varepsilon}{1 + (2 \|f\|_{L^\infty})^{(p-2)/p}} \cdot (2 \|f\|_{L^\infty})^{(p-2)/p} \leq \varepsilon.
\end{aligned}$$

Hence the sequence is Cauchy and therefore converges in L^p whenever $f \in L^p \cap L^\infty$. To conclude, we exploit the density we proved earlier. In fact, given $f \in L^p$, we may choose $g_k \in L^p \cap L^\infty$ such that $\|f - g_k\|_{L^p} \to 0$ as $k \to \infty$. Then, letting $\varepsilon > 0$ and k large enough such that $\|f - g_k\|_{L^p} < \varepsilon/4$, we may choose M, N large enough such that

$$\left\| \frac{1}{M} \sum_{j=0}^{M-1} g_k \circ T^{-j} - \frac{1}{M} \sum_{j=0}^{M-1} g_k \circ T^{-j} \right\|_{L^p} < \varepsilon/2$$

since $g_k \in L^p \cap L^\infty$. Then the averages for f are seen to form a Cauchy sequence in L^p since

$$\begin{aligned}
&\left\| \frac{1}{N} \sum_{j=0}^{N-1} f \circ T^{-j} - \frac{1}{M} \sum_{j=0}^{M-1} f \circ T^{-j} \right\|_{L^p} \\
&\leq \left\| \frac{1}{N} \sum_{j=0}^{N-1} f \circ T^{-j} - \frac{1}{N} \sum_{j=0}^{N-1} g_k \circ T^{-j} \right\|_{L^p} + \left\| \frac{1}{N} \sum_{j=0}^{N-1} g_k \circ T^{-j} - \frac{1}{M} \sum_{j=0}^{M-1} g_k \circ T^{-j} \right\|_{L^p} \\
&\qquad + \left\| \frac{1}{M} \sum_{j=0}^{M-1} g_k \circ T^{-j} - \frac{1}{M} \sum_{j=0}^{M-1} f \circ T^{-j} \right\|_{L^p} \\
&\leq \left\| \frac{1}{N} \sum_{j=0}^{N-1} (f - g_k) \circ T^{-j} \right\|_{L^p} + \left\| \frac{1}{M} \sum_{j=0}^{M-1} (f - g_k) \circ T^{-j} \right\|_{L^p} \\
&\qquad + \left\| \frac{1}{M} \sum_{j=0}^{M-1} g_k \circ T^{-j} - \frac{1}{M} \sum_{j=0}^{M-1} g_k \circ T^{-j} \right\|_{L^p} \\
&\leq 2 \|f - g_k\|_{L^p} + \varepsilon/2 \leq \varepsilon.
\end{aligned}$$

We conclude by completeness of L^p and because $2 \leq p < \infty$ was arbitrary. □

Problem 8. Let $A = \begin{pmatrix} -1 & -1 \\ 1 & 2 \end{pmatrix}$. Show that the map $x \mapsto Ax$ from $\mathbb{R}^2$ to itself induces a map

$$T : \mathbb{R}^2/\mathbb{Z}^2 \longrightarrow \mathbb{R}^2/\mathbb{Z}^2.$$

Is this map transitive? Determine the global stable and unstable manifolds $W^{\pm}$ of $0 \in \mathbb{R}^2/\mathbb{Z}^2$, where

$$W^{\pm}(0) = \{p \in \mathbb{R}^2/\mathbb{Z}^2,\, T^{\pm n}p \longrightarrow 0\}.$$

Solution. We begin by diagonalizing the problem in $\mathbb{R}^2$. Notice that since it is linear, the conjugating homeomorphism given by the Hartman-Grobman theorem will be a simple change of basis in the vector space structure of $\mathbb{R}^2$, and is in fact global. The eigenvalues of the matrix A are

$$\lambda_- = \frac{1+\sqrt{5}}{2} > 1, \quad \lambda_+ = \frac{1-\sqrt{5}}{2} \in (-1, 0),$$

so the system is hyperbolic. The stable and unstable manifolds are spanned as

$$W^+_{\mathbb{R}^2} = \ker(A - \lambda_+ I) = \left\langle \begin{pmatrix} 1 \\ -\frac{3-\sqrt{5}}{2} \end{pmatrix} \right\rangle, \quad W^-_{\mathbb{R}^2} = \ker(A - \lambda_- I) = \left\langle \begin{pmatrix} 1 \\ -\frac{3+\sqrt{5}}{2} \end{pmatrix} \right\rangle$$

respectively. Notice that without passing to the quotient space, the map is not transitive, as any point in $\mathbb{R}^2$ in the stable manifold cannot have a dense orbit, and any point with a nonzero unstable projection will eventually escape along a direction approaching that of the unstable manifold, making it impossible for a dense orbit to appear.

On the other hand, it is transitive once we look at it on the torus. To show its projection is well defined, notice that if $(x, y) - (z, t) = (n, m) \in \mathbb{Z}^2$, then

$$\begin{aligned} T(x, y) - T(z, t) &= (-x - y, x + 2y) - (-z - t, z + 2t) \mod 1 \\ &= (-n - m, n + 2m) \mod 1 = (0, 0) = T(n, m). \end{aligned}$$

The stable and unstable manifolds simply take the form of the projections, so

$$\begin{aligned} W^+(0) &= \left\{ (x, y) \in [0, 1)^2 \colon y = -\frac{3-\sqrt{5}}{2}x \mod 1 \right\}, \\ W^-(0) &= \left\{ (x, y) \in [0, 1)^2 \colon y = -\frac{3+\sqrt{5}}{2}x \mod 1 \right\}. \end{aligned}$$

Moreover, they are dense sets in $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Q}^2$. Indeed, write

$$\alpha_+ = -\frac{3-\sqrt{5}}{2}, \quad \alpha_- = -\frac{3+\sqrt{5}}{2}.$$

For the unstable manifold, given any point $(x, y) \in \mathbb{T}^2$ and $\varepsilon > 0$ it suffices to find a line of slope α_- connecting $B_\varepsilon(x, y)$ with the line $y = 0$. Then, density of $W^-(0)$ in $\{y = 0\}$ grants the existence of a point in the unstable manifold that is arbitrarily close to where the aforementioned line intersects $\{y = 0\}$.

The fact that $W^-(0)$ is dense in $y = 0$ amounts to, given $x \in (0, 1)$ and $\varepsilon > 0$, finding $k, m \in \mathbb{Z}$ such that

$$|k\alpha_-^{-1} + m - x| < \varepsilon. \tag{4}$$

Let $R_\alpha(z)$ be the rotation in the unit circle $\mathbb{S}^1$, interpreted on $\mathbb{T}^1$ (that is, $R_\alpha(z) = z + \alpha \mod 1$). Then, observe that (4) is the same as asking for $k \in \mathbb{Z}$ such that $R^k_{\alpha_-^{-1}}(0) \in B_\varepsilon(x)$, but this is satisfied for some k immediately, since any orbit is dense whenever α (in this case α_-^{-1}) is irrational (this is **Proposition 2.6 (ii)** in Lecture1.pdf). To deduce the analogous result for the stable manifold, we may consider T^{-1} and use the same argument.

Now we deal with the transitivity of T. For this, we will prove that T is ergodic and that this implies transitivity. To start with, notice that the determinant of A is equal to 1, which implies that T is measure-preserving (we are using **Lemma 1.2** from Lecture4.pdf). Next, we use the fact that T being ergodic is equivalent to the only functions $f \in L^2$ satisfying $f(T(x)) = f(x)$ being the constant ones.

Claim

Any linear hyperbolic automorphism $\varphi\colon \mathbb{T}^2 \longrightarrow \mathbb{T}^2$ of determinant ± 1 is ergodic. Suppose that φ is given by the matrix $A \in \mathrm{Mat}(2 \times 2, \mathbb{Z})$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

and assume that $f \in L^2$ is invariant, $f \circ \varphi^k = f$. Then, by looking at this equality on the Fourier level (computing the Fourier coefficients of both sides), we easily reach

$$\widehat{f}(m,n) = \widehat{f \circ \varphi}(n,m) = \widehat{f}\,(\det A(md - nc), \det A(na - mb)) = \widehat{f}((m,n) \cdot A^{-1}),$$

with a slight abuse of notation. Iterating, $\widehat{f}(m,n) = \widehat{f}((m,n) \cdot A^{-k})$.
Denote $v = (m,n)$, and notice that for vA^{-k} to have a periodic orbit in k it would be necessary that $(A^{-k})^T$ had 1 for an eigenvalue, which violates the hyperbolicity of A unless $v = 0$. On the other hand, if vA^{-k} does not sit on a periodic orbit, then it must happen that

$$|v \cdot A^{-k}| \longrightarrow \infty, \quad \text{if } |(m,n)| \to \infty.$$

Now since $\left\{\widehat{f}(n,m)\right\}_{n,m} \subset L^2$, we must have convergence to zero of the Fourier coefficients if $|(m,n)| \to \infty$. But then for any $(m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\}$, we must have

$$\widehat{f}(n,m) = \lim_{k\to\infty} \widehat{f}(v \cdot A^{-k}) = 0.$$

This means that the only surviving coefficient is $(m,n) = 0$, which precisely encodes that f must be a constant function $f = \widehat{f}(0,0)$.

The claim shows that T is ergodic, and it remains to check that this implies transitivity. Assume for contradiction that T is not transitive. Then, there are nonempty, open sets $U, V \subset \mathbb{T}^2$ for which $T^n(U) \cap V = \emptyset$ for all $n \geq 1$. Define

$$W = \bigcup_{n \geq 0} T^{-n}(U),$$

which is easily checked to be invariant, and notice that $W \cap V = \emptyset$, and that V is of positive measure. Since U is as well, it follows that the measure of W sits in $(0,1)$, but because it is T-invariant, this contradicts the ergodicity of T. □

Notes

We state here the results we referred to in the solutions.

Theorem (Birkhoff transitivity)**.** *Let* (X, d) *be a complete metric space with a countable basis of open sets, and* $T : X \longrightarrow X$ *a continuous map. If for every pair of nonempty open sets* $U, V \subset X$ *there is* $n \in \mathbb{N} \cup \{0\}$ *such that* (1) *holds true, then* T *is transitive. In fact, the set of points with a dense orbit is dense.*

Lemma (Lemma 1.2, Lecture4.pdf)**.** *Let* (X, m) *be a measure space. Then* $T \colon X \longrightarrow X$ *is measure preserving if and only if*

$$\int_X f \circ T \, \mathrm{d}m = \int_X f \, \mathrm{d}m$$

for all $f \in L^1(X, m)$.

Lemma (Proposition 2.6 (ii), Lecture1.pdf)**.** *If* $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ *is irrational, then every orbit of* $\phi(z) = e^{2\pi i \alpha} z$ *is dense.*

In **Problem 1**, the transitivity of T can be approached explicitly by providing an element with a dense orbit. This gets very technically involved after finding the correct construction and escapes the scope of the course.

Problem 2 can also be approached through a different method, by noting that the map T corresponds to the doubling map,

$$T(\cot \theta) = \cot(2\theta).$$

Transitivity in **Problem 8** can be deduced from the fact that T is also **mixing**. For this, it is necessary to argue using **Lemma 2.2** in Lecture7.pdf on the basis of $L^2(\mathbb{T}^2)$.